ASYMPTOTIC SOLUTION OF A SUBLIMATION PROBLEM

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On the subliming surface of a semi-infinite solid body the temperature T_S = const and the gradient $\varphi(t)$ are given, together with the temperature at infinity T_0 = const and the initial temperature distribution. These conditions are sufficient for finding the temperature distribution $T(x, t)$ and the linear rate of sublimation when $t > 0$. In a system of coordinates related to the surface, the problem has the form

$$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2} + u\frac{\partial T}{\partial x}, \quad T(0, t) = T_s,$$
$$-\frac{\partial T}{\partial x}(0, t) = \varphi, \quad T(\infty, t) = T_0. \tag{1}$$

The initial condition has not been written in (1) only because it is not required for the asymptotic solution.

For the case φ = const, the Hertz-Michelson solution is well-known:

$$\frac{T - T_0}{T_s - T_0} = \exp\left(-\frac{u}{a}x\right), \quad u = \frac{a\varphi}{T_s - T_0}.$$

In the case of slowly varying φ it is convenient to introduce the notation

$$u_\varphi(t) = \frac{a\varphi(t)}{T_s - T_0}, \quad \varepsilon = \frac{T - T_0}{T_s - T_0} - \exp\left(-\frac{u_\varphi}{a}x\right),$$
$$\eta = \int_0^t \frac{u_\varphi^2}{a}dt, \quad \xi = \frac{u_\varphi}{a}x.$$

Then (1) takes the form

$$\frac{\partial\varepsilon}{\partial\eta} + \left(\alpha\xi - \frac{u}{u_\varphi}\right)\frac{\partial\varepsilon}{\partial\xi} - \frac{\partial^2\varepsilon}{\partial\xi^2} = \left(\alpha\xi - \frac{u}{u_\varphi} + 1\right)\exp(-\xi), \quad \varepsilon(0, \eta) = \frac{\partial\varepsilon}{\partial\xi}(0, \eta) = \varepsilon(\infty, \eta) = 0, \tag{2}$$

$$\frac{u}{u_\varphi} = 1 + \alpha + \int_0^\infty\left(\alpha\varepsilon - \frac{\partial\varepsilon}{\partial\eta}\right)d\xi, \quad \alpha = \frac{d\ln u_\varphi}{d\eta}.$$

The expression for u/u_φ was obtained by integrating the equation with respect to ξ from 0 to ∞.

We shall seek ε in the form

$$\varepsilon = \exp(-\xi)\sum_{k=2}^{\infty} c_k(\eta)\frac{\xi^k}{k!}. \tag{3}$$

Substituting (3) into (2), and equating coefficients of like powers of ξ, we obtain a system of equations for c_k:

$$c_2 + 1 - u/u_\varphi = 0, \quad c_2(2 - u/u_\varphi) - c_3 - \alpha = 0,$$
$$c_k' + c_k(\alpha k + u/u_\varphi - 1) - \alpha c_{k-1} + k(2 - u/u_\varphi)c_{k+1} - k(k+1)c_{k+2} = 0, \tag{4}$$
$$u/u_\varphi = 1 + \alpha + \sum_{k=2}^{\infty}(\alpha c_k - c_k').$$

For slow variation of φ, system (4) is equivalent to a system with a small parameter in the presence of derivatives.

As regards the argument t, this role will be played by the quantity a/u_φ^2, which must be small in comparison with the characteristic time of variation of φ or u_φ:

$$\frac{a}{u_\varphi^2}\frac{d\ln u_\varphi}{dt} = \alpha \ll 1.$$

We may seek an asymptotic solution when $a/u_\varphi^2 \to 0$ in the form of a power series in a/u_φ^2. In dimensionless variables this will simply be a series in the derivatives, which we find by successive approximations. In the second approximation, restricting attention to quantities of the order α^2 and α', we have

$$c_2 \approx \alpha + (\alpha^2 - \alpha'), \quad c_3 \approx -\alpha', \quad c_k \approx 0, \quad k > 3.$$

Then the last formula of (4) gives

$$u/u_\varphi = 1 + \alpha + (\alpha^2 - \alpha') + (\alpha^3 - 4\alpha\alpha' + 2\alpha'') + \ldots,$$

or in dimensional variables

$$\frac{u}{u_\varphi} = 1 + \frac{a}{u_\varphi^2}\left(\frac{d\ln u_\varphi}{dt}\right) + \left(\frac{a}{u_\varphi^2}\right)^2\left[3\left(\frac{d\ln u_\varphi}{dt}\right)^2 - \frac{d^2\ln u_\varphi}{dt^2}\right] + \left(\frac{a}{u_\varphi^2}\right)^3\left[9\left(\frac{d\ln u_\varphi}{dt}\right)^3 - 20\frac{d\ln u_\varphi}{dt}\frac{d^2\ln u_\varphi}{dt^2} + 2\frac{d^3\ln u_\varphi}{dt^3}\right] + \ldots. \tag{5}$$

To any accuracy of order α, Eq. (5) is well-known as the Zel'dovich formula.

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